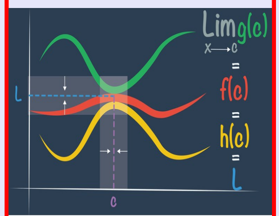
Calculus I

Lecture 14



Feb 19-8:47 AM

Class QZ 11

Given $x^3+y^3=2$ **open notes**

1) Find y' and evaluate y' at $(1,1)$.

$3x^2+3y^2y'=0 \qquad 3y^2y'=-3x^2 \qquad y'=\frac{-\cancel{3}x^2}{\cancel{3}y^2} \qquad \boxed{y'=\frac{-x^2}{y^2}}$

$y'=\frac{dy}{dx}\Big|_{(1,1)}=\frac{-1^2}{1^2}=-\frac{1}{1}=\boxed{-1}$

2) Find y'' and evaluate y'' at $(1,1)$.

$y''=\frac{d}{dx}\left[y'\right]=\frac{d}{dx}\left[-\frac{x^2}{y^2}\right]=-\frac{d}{dx}\left[\frac{x^2}{y^2}\right]=-\frac{2xy^2-x^2\cdot 2yy'}{(y^2)^2}$

$=-\frac{2xy^2-2x^2yy'}{y^4}=-\frac{\cancel{y}(2xy-2x^2y')}{y^{\cancel{4}\,3}}=\boxed{\frac{2x^2y'-2xy}{y^3}}$

$\frac{d^2y}{dx^2}\Big|_{(1,1)}=\frac{2(1)^2(-1)-2(1)(1)}{1^3}=\frac{-2-2}{1}=\boxed{-4}$ ✓

Jul 9-7:39 AM

$$y'' = \frac{2x^2y' - 2xy}{y^3} \qquad y'' = \frac{2x^2 \cdot \frac{-x^2}{y^2} - 2xy}{y^3}$$

$$LCD = y^2$$

$$y'' = \frac{-2x^4 - 2xy^3}{y^3 \cdot y^2} = \frac{-2x(x^3+y^3)}{y^5}$$

$$= \frac{-2x \cdot 2}{y^5} = \frac{-4x}{y^5}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(1,1)} = \frac{-4(1)}{1^5} = \frac{-4}{1} = \boxed{-4} \qquad y'' = \frac{-4x}{y^5}$$

Jul 9-8:30 AM

Given $\sqrt[3]{x^2} - \sqrt[3]{y^2} = 3$ $\qquad \sqrt[n]{x^m} = x^{\frac{m}{n}}$

find $\frac{dy}{dx}$, evaluate $\left.\frac{dy}{dx}\right|_{(8,1)}$.

$$x^{2/3} - y^{2/3} = 3$$

$$\frac{2}{3}x^{-1/3} - \frac{2}{3}y^{-1/3}y' = 0$$

Multiply by $\frac{3}{2}$

$$x^{-1/3} - y^{-1/3}y' = 0$$

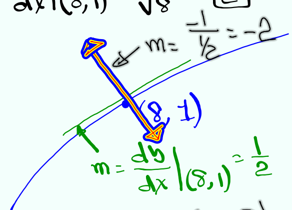
$$-y^{-1/3}y' = -x^{-1/3}$$

$$y' = \frac{-x^{-1/3}}{-y^{-1/3}}$$

$$x^{-n} = \frac{1}{x^n} \qquad \frac{x^{-n}}{y^{-m}} = \frac{y^m}{x^n}$$

$$y' = \frac{y^{1/3}}{x^{1/3}} = \frac{\sqrt[3]{y}}{\sqrt[3]{x}}$$

$$\left.\frac{dy}{dx}\right|_{(8,1)} = \frac{\sqrt[3]{1}}{\sqrt[3]{8}} = \boxed{\frac{1}{2}}$$



$$y - 1 = \frac{1}{2}(x-8)$$

$\boxed{y = \frac{1}{2}x - 3}$ eqn of tan. line

$$y - 1 = -2(x-8)$$

$\boxed{y = -2x + 17}$ eqn of Normal line

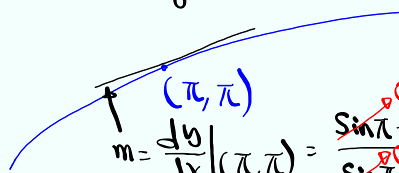
Jul 9-8:34 AM

Find eqn of the tan. line to the graph of

$y \sin x = x \sin y$ at (π, π).

$y - \pi = 1(x - \pi)$

$\boxed{y = x}$



$$m = \left.\frac{dy}{dx}\right|_{(\pi,\pi)} = \frac{\sin\pi - \pi\cos\pi}{\sin\pi - \pi\cos\pi} = \frac{0 - \pi(-1)}{0 - \pi(-1)} = \frac{\pi}{\pi} = 1$$

$$\frac{dy}{dx}\sin x + y\cos x = 1 \cdot \sin y + x\cos y\frac{dy}{dx}$$

$$\frac{dy}{dx}\sin x - x\cos y\frac{dy}{dx} = \sin y - y\cos x$$

$$(\sin x - x\cos y)\frac{dy}{dx} = \sin y - y\cos x$$

$$\frac{dy}{dx} = \frac{\sin y - y\cos x}{\sin x - x\cos y}$$

Jul 9-8:46 AM

Estimate $\frac{1}{\sqrt{5}}$

By Calculator $\rightarrow \approx .447$

$$\frac{1}{\sqrt{5}} \approx \frac{1}{\sqrt{4}} = \frac{1}{2}$$

$f(x) = \frac{1}{\sqrt{x}}$

$a = 4$

Linear Approximation

$$\mathbf{f(x) \approx f(a) + f'(a)(x - a)}$$

$f(x) = \frac{1}{x^{1/2}}$

$f(x) = x^{-1/2}$

$f'(x) = -\frac{1}{2}x^{-3/2}$

$f'(x) = \frac{-1}{2x^{3/2}}$

$$\frac{1}{\sqrt{x}} \approx \frac{1}{\sqrt{4}} + f'(4)(x - 4)$$

$$\approx \frac{1}{2} + \frac{-1}{16}(x - 4)$$

$$\frac{1}{\sqrt{5}} \approx \frac{1}{2} - \frac{1}{16}(5 - 4)$$

$$= \frac{1}{2} - \frac{1}{16} = \frac{8 - 1}{16} = \frac{7}{16} = \boxed{.4375}$$

$$f'(4) = \frac{-1}{2 \cdot 4^{3/2}} = \frac{-1}{2 \cdot 8} = \frac{-1}{16}$$

$$4^{3/2} = \sqrt{4^3} = (\sqrt{4})^3 = 2^3 = 8$$

Jul 9-8:57 AM

Estimate $(5.1)^4$

By Calculator $\to$ 676.5201

Common Sense $5.1^4 \approx 5^4 = 625$

Linear Approximation

$f(x) = x^4$

$a = 5$

$f(a) = 5^4 = 625$

$f'(x) = 4x^3$

$f'(5) = 4(5)^3 = 4(125)$
$= 500$

$f(x) \approx f(a) + f'(a)(x-a)$

$x^4 \approx 625 + 500(x-5)$

$5.1^4 \approx 625 + 500(5.1-5)$

$= 625 + 500(.1)$

$= 625 + 50 = \boxed{675}$

Jul 9-9:07 AM

Estimate $\sqrt[4]{15}$

using Calculator $\to \sqrt[4]{15} \approx 1.968$

Common Sense $\sqrt[4]{15} \approx \sqrt[4]{16} = 2$

Linear Approx.

$f(x) = \sqrt[4]{x}$

$a = 16$

$f(16) = \sqrt[4]{16} = 2$

$f(x) = x^{1/4}$

$f'(x) = \frac{1}{4}x^{-3/4}$

$= \frac{1}{4\sqrt[4]{x^3}} = \frac{1}{4(\sqrt[4]{x})^3}$

$f'(16) = \frac{1}{4(\sqrt[4]{16})^3} = \frac{1}{4 \cdot 8} = \frac{1}{32}$

$f(x) \approx f(a) + f'(a)(x-a)$

$\sqrt[4]{x} \approx 2 + \frac{1}{32}(x-16)$

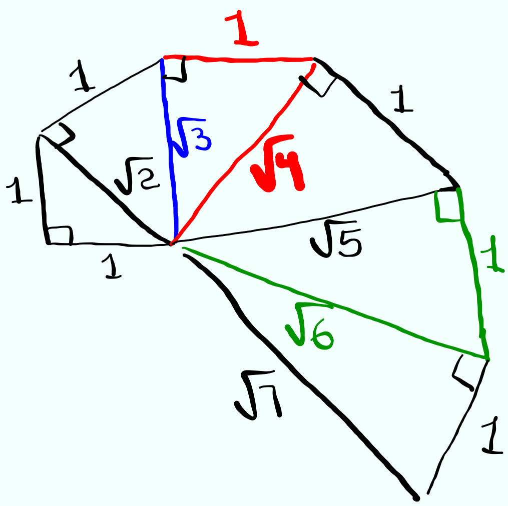
$\sqrt[4]{15} \approx 2 + \frac{1}{32}(15-16)$

$= 2 + \frac{1}{32} \cdot (-1) = 2 - \frac{1}{32}$

$= \frac{63}{32}$

$= \boxed{1.96875}$

Jul 9-9:16 AM



Jul 9-9:30 AM

More on related rates:

Given $x^2 + y^2 = z^2$

$4^2 + 5^2 = z^2$

$16 + 25 = z^2$

$41 = z^2$

Find $\frac{dz}{dt}$ when $x = 4$, $\frac{dx}{dt} = -3$, $y = 5$, and $\frac{dy}{dt} = 2$.

$z = \sqrt{41}$

$$\frac{d}{dt}[x^2] + \frac{d}{dt}[y^2] = \frac{d}{dt}[z^2]$$

$$\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} = \cancel{2}z\frac{dz}{dt}$$

Divide by 2

$$4(-3) + 5(2) = \sqrt{41}\,\frac{dz}{dt}$$

$$-2 = \sqrt{41}\,\frac{dz}{dt}$$

$$\frac{dz}{dt} = \frac{-2}{\sqrt{41}} = \frac{-2\sqrt{41}}{41}$$

x decreasing, $\frac{dx}{dt} < 0$

y increasing, $\frac{dy}{dt} > 0$

z decreasing

$\frac{dz}{dt} < 0$

Jul 9-9:56 AM

Snowball is melting at the rate of 1 cm³/min. $\frac{dV}{dt}=-1$

How fast its diameter changing when diameter is 5 cm? $d=2r$

$r=\frac{d}{2}$

Sphere

$V=\frac{4\pi r^3}{3}$ $\quad V=\frac{4\pi}{3}\left(\frac{d}{2}\right)^3$

$V=\frac{\cancel{4}\pi d^3}{3\cdot\cancel{8}_2}$ $\quad V=\frac{\pi d^3}{6}$

$\frac{dV}{dt}=\frac{\pi}{\cancel{6}_2}\cdot\cancel{3}d^2\frac{dd}{dt}$

$-1=\frac{\pi}{2}(5)^2\frac{dd}{dt}$

$-1=\frac{25\pi}{2}\frac{dd}{dt}$

$\frac{dd}{dt}=\frac{-2}{25\pi}$ cm/min.

Jul 9-10:03 AM

The height of a triangle increasing at 2 cm/min. $\frac{dh}{dt}=2$ cm/min.

The area of that triangle decreases at 3 cm²/min. $\frac{dA}{dt}=-3$ cm²/min

How fast is the base changing when height is 10 cm and area is 100 cm²?

Area formula for triangle $\quad \frac{db}{dt}$? $h=10$, $A=100$

$A=\frac{bh}{2}$ $\quad A=\frac{bh}{2}$

$2A=bh$ $\quad 100=\frac{b\cdot 10}{2}$

$\frac{d}{dt}[2A]=\frac{d}{dt}[bh]$ $\quad 100=5b$

$b=20$

$2\frac{dA}{dt}=\frac{db}{dt}\cdot h+b\cdot\frac{dh}{dt}$

$2(-3)=\frac{db}{dt}(10)+20\cdot 2$

$-6-40=10\frac{db}{dt}$

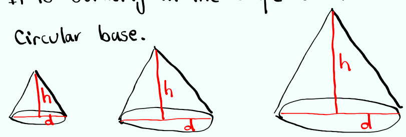
$\frac{db}{dt}=\frac{-46}{10}=-4.6$ cm/min.

Jul 9-10:09 AM

Gravel is being dumped at a rate of 30 ft³/min

$\frac{dV}{dt} = 30$

It is forming in the shape of cone with circular base.



Now it is always height = diameter.

How fast the height is increasing when the pile is $\boxed{\text{10 ft high}}$?

Volume of a Cone

$2r = d$

$r = \frac{d}{2}$

$d = h$

$\boxed{r = \frac{h}{2}}$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h$

$V = \frac{\pi}{3} \frac{h^3}{4}$

$V = \frac{\pi}{12} h^3$

$\frac{dV}{dt} = \frac{\pi}{\cancel{12}_4} \cdot \cancel{3} h^2 \frac{dh}{dt}$

$30 = \frac{\pi}{4}(10)^2 \frac{dh}{dt}$

$120 = 100\pi \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{\cancel{120}^6}{\cancel{100}_5 \pi}$

$\boxed{\frac{dh}{dt} = \frac{6}{5\pi} \text{ ft/min.}}$

Jul 9-10:19 AM

A 10-ft ladder is leaning against a vertical wall.

$x^2 + y^2 = 10^2$

The base of the ladder is being pulled away from the wall at the rate of 3 ft/min.

x increasing $\frac{dx}{dt} = 3$

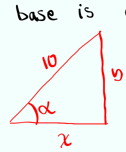
How fast is the angle between the ladder and the ground is changing when the base is 6 ft from the wall?

$\frac{d\alpha}{dt} = ?$ $x = 6$

$x^2 + y^2 = 10^2$

$6^2 + y^2 = 100$

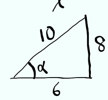
$y = 8$



$\sin\alpha = \frac{y}{10}$

$\boxed{\cos\alpha = \frac{x}{10}}$

$\tan\alpha = \frac{y}{x}$



$\sin\alpha = \frac{8}{10}$

$10\cos\alpha = x$

$\frac{d}{dt}[10\cos\alpha] = \frac{d}{dt}[x]$

$10 \cdot -\sin\alpha \cdot \frac{d\alpha}{dt} = \frac{dx}{dt}$

$-\cancel{10} \cdot \frac{8}{\cancel{10}} \frac{d\alpha}{dt} = 3$

$-8 \frac{d\alpha}{dt} = 3$

$\frac{d\alpha}{dt} = \frac{-3}{8}$ Rad/min.

Decreasing

Jul 9-10:30 AM

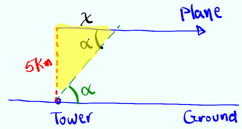
A plane is flying horizontally.

It passes a tower on the ground.

Plane is flying at an altitude of 5km.

when the angle of elevation from the tower to the plane is $\frac{\pi}{3}$, angle is decreasing at $\frac{\pi}{6}$ Rad./min. $\frac{d\alpha}{dt} = -\frac{\pi}{6}$

How fast is the plane flying?



$\sin\alpha = ?$

$\cos\alpha = ?$

$\tan\alpha = \frac{5}{x}$

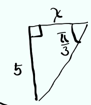
Cross-Multiply

$x \tan\alpha = 5$

$\frac{dx}{dt} \tan\alpha + x \cdot \sec^2\alpha \cdot \frac{d\alpha}{dt} = 0$

$\frac{dx}{dt} \cdot \tan\frac{\pi}{3} + \frac{5}{\sqrt{3}} \cdot \sec^2\frac{\pi}{3} \cdot \frac{-\pi}{6} = 0$

$\frac{dx}{dt} \cdot \sqrt{3} + \frac{5}{\sqrt{3}} \cdot 2^2 \cdot \frac{-\pi}{6} = 0$



$\tan\frac{\pi}{3} = \frac{5}{x}$

$\sqrt{3} = \frac{5}{x}$

$x = \frac{5}{\sqrt{3}}$

$\sqrt{3}\,\frac{dx}{dt} = \frac{5}{\sqrt{3}} \cdot \frac{4\pi}{6}$

$\frac{dx}{dt} = \frac{\frac{20\pi}{6\sqrt{3}}}{\sqrt{3}} = \frac{20\pi}{6 \cdot 3} = \frac{20\pi}{18} = \frac{10\pi}{9}$ Km/min

Speed per hr $\rightarrow \frac{10\pi}{9} \cdot 60$ Km/hr.

$\approx \boxed{210 \text{ Km/hr.}}$

Jul 9-10:44 AM

I will give you

$f(x)$, $f'(x)$, and $f''(x)$

1) Domain of $f(x)$

2) x-Values where $f'(x) = 0$ or undefined.

3) x-Values where $f''(x) = 0$ or undefined.

Jul 9-11:04 AM